Mathematics as the Source of Music Composition

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This paper describes three of numerous projects in which I have composed music by the use of mathematical formulas. It is much less a mathematical treatise than an informal indication of solutions I chose and reasons I chose them in order to compose.

In 1970 I wrote the composition *Cheltrovype* for chamber ensemble, in which I for the first time generated a series of pitches by means of probability calculations. For the choice of every attack, the probability of every note in the individual instruments' ranges was computed according to a set of formulas and the result was generated by the use of random numbers.

1978 saw the beginnings of my system of quantification of harmony and meter for the generation of melodies and rhythms as implemented in my computer program *Autobusk* (1986-2002). Here I calculated the probability of attack of every pulse in a given meter or meters, and if there was to be an attack, the probability of all the pitches (in a given scale or scales) of which one was to be played. These probabilities were derived from the relative metric importance of each pulse and the relative harmonic importance of each pitch, based on formulas I developed for the purpose.

In 2001 I introduced my digital sound processing technique *ISIS*, whereby the interpolation of imaginary connecting sinusoids between the samples of a sound wave yields possibilities for converting the latter into a melody and vice versa. It is possible by these means to view any sound wave as a microtonal sequence of sine tones running at the sampling rate and centered in pitch at the sampling rate frequency. This 'melody' can be slowed down to a perceptible speed and transposed down to the audible frequency range. In reverse, any melody can be converted into a sound wave. The domains of rhythm, pitch and timbre here form a continuum.

Keywords: probability, mode, meter, timbre, sinusoid, harmonicity.

Cheltrovype

In 1970 my composition teacher Bernd Alois Zimmermann gave me the task of writing a piece for 'cello, trombone, vibraphone and percussion. I finished it as a piece entitled *Cycle* just three months before his untimely death in August of that year, but he was sadly too ill to see it. In June I decided to make *Cycle* Part III of a larger work entitled *Cheltrovype* (see the instrumentation) comprising six movements. This description concerns Part V, completed in March 1971.

As I imagined, the 'cello would begin this movement with a series of almost regular, fairly rapid notes on its lowest note C_2. Gradually notes chromatically higher – D-flat$_2$, D_2 etc. – would be introduced, the lowest note always remaining C_2. The range would finally increase to about 3½ octaves, the most frequent notes shifting upwards in pitch, with the movement ending on F_5. The trombone would start the same process on C_3 a while after the 'cello entry, gradually rising to include and then end on F_5 together with the 'cello. The vibraphone would start on $C_{4,}$ later than the trombone entry, and end together with the 'cello and trombone on F_5.

My first attempts to write this music were not successful. How many initial Cs should occur before the first D-flat? It did not take long before I realized that I would have to formalize the process algebraically and determine the pitches by means of probability calculations and random numbers. See Fig.1 for an illustration of the lowest, highest and most frequent pitches against time. These variables are shown as algebraic formulas and curves: the x-axis shows time, the y-axis shows MIDI pitch numbers. A fourth variable, the relative probability of all pitches at any time, a distorted parabola, extends into the third dimension (z-axis) and is not shown here.

Fig.1 – three pitch-time variables

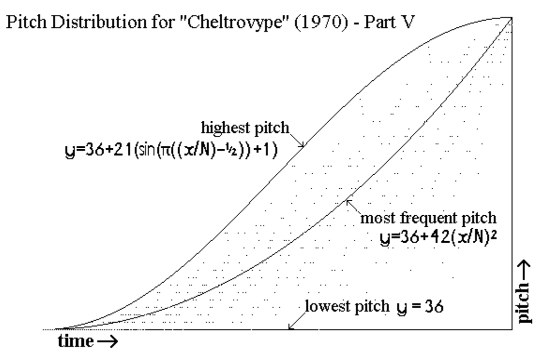


Fig.2 – the resultant 'cello part

Given that the 'cello part would contain 500 notes over a range of 42 semitones (C_2-F_5), I first tried to calculate the probability values – 21,000 for the 'cello alone – using logarithmic tables (this was 1970), a hopeless task. I then tried a 50 lb electromechanical calculator (equipped with a handle in case of the lack of electricity); that's when I realized that a computer would be ideal for the job. Accordingly I learned Fortran at Cologne University and had all probability tables printed out within a week of starting the course. Then my electromechanical home calculator generated 'random' numbers. Fig.2 shows the 'cello part. This was one of the very first scores of music (certainly the first to my knowledge) generated in Germany by a computer.

Autobusk

In 1975 while traveling in Turkey, inspired by the music I heard there, I decided to write a large piano piece using quarter-tones. Being equally at home in the Western tonal and metric music of the past few centuries and the Western atonal and ametric music of Post-World War II, I also wished to generate tonal and metric fields of variable strength in this piece, i.e. music in a continuum ranging freely between the tonal and the atonal, the metric and the ametric. This I sought to achieve by setting up probability tables for the choice of pulses to attack and pitches to attack them with. The probabilities would be based on a list of tonal and metric priorities for the pitches and pulses in given modes and meters. Had I chosen to use pitches known to Western Music and its theoreticians like Hindemith, I could simply have derived my priorities from historic rankings of the relative harmonic strength (or 'harmonicity', as I call it) of the intervals in my modes. However, I had my eye on scales containing quarter-tones, of which no harmonic evaluation or pitch ranking was known to me.

I accordingly studied various attempts to establish the harmonicity of an interval by scholars reaching back through Partch, Euler (e.g. his 'Totient Function') all the way to Pythagoras and Indian theorists of three centuries thereafter. The main idea behind all these hypotheses was that the smaller the numbers forming the ratio of an interval of pitch, the more harmonic that interval. I couldn't help noticing that whereas many intervals of the form $n:n+1$ (e.g. 1:2, 2:3 upto 15:16) are common in classical European and Indian music, the series is interrupted by the lack in both of the 6:7, 7:8, 10:11, 11:12, 12:13, 13:14 and 14:15, intervals containing prime numbers larger than 5. It was therefore clear to me that a coefficient pertaining to both the smallness and the divisibility of a number (i.e. the smallness of the prime factors contained therein) was necessary, which after some research I developed in the form of my function of the 'indigestibility' of a number – the smaller and more divisible a number, the less indigestible it would be. This function I then used to develop a formula for the harmonicity of an interval expressed as a ratio. These functions are seen as formulas in Fig.3 and tables in Fig.4. Notice the high indigestibility of the larger primes.

Fig.3 – Indigestibility, Harmonicity Formulas

Indigestibility Formula

$$\xi(N) = 2\sum_{r=1}^{\infty}\left\{\frac{n_r(p_r-1)^2}{p_r}\right\}$$

whereby:

1. $N = \prod_{r=1}^{\infty} p_r^{n_r}$
2. $N,\ n,\ p$ $\in$ natural numbers
3. p $\in$ prime numbers

Harmonicity Formula

$$H(P,Q) = \frac{\left(\xi(Q)-\xi(P)\right)}{\xi(P)+\xi(Q)}$$

whereby $\mathrm{sgn}(x)=-1$ for $x<0$, else $\mathrm{sgn}(x)=+1$

Fig.4 – Indigestibility and Harmonicity Tables

Indigestibility ξ(N) of the natural numbers 1–16

N	ξ(N)
1	0.000000
2	1.000000
3	2.666667
4	2.000000
5	6.400000
6	3.666667
7	10.285714
8	3.000000
9	5.333333
10	7.400000
11	18.181818
12	4.666667
13	22.153846
14	11.285714
15	9.066667
16	4.000000

Complete Intraoctavic Intervals upwards of Harmonicity 0.05

Interval-size (Ct)	Prime Decomposition as Powers of 2	3	5	7	11	13	Number-ratio	Harmonicity
0.000	0	0	0	0	0	0	1:1	+∞
70.672	−3	−1	+2	0	0	0	24:25	+0.054152
111.731	+4	−1	−1	0	0	0	15:16	−0.076531
182.404	+1	−2	+1	0	0	0	9:10	+0.078534
203.910	−3	+2	0	0	0	0	8:9	+0.120000
231.174	+3	0	0	−1	0	0	7:8	−0.075269
266.871	−1	−1	0	+1	0	0	6:7	+0.071672
294.135	+5	−3	0	0	0	0	27:32	−0.076923
315.641	+1	+1	−1	0	0	0	5:6	−0.099338
386.314	−2	0	+1	0	0	0	4:5	+0.119048
407.820	−6	+4	0	0	0	0	64:81	+0.060000
427.373	+5	0	−2	0	0	0	25:32	−0.056180
435.084	0	+2	0	−1	0	0	7:9	−0.064024
470.781	−4	+1	0	+1	0	0	16:21	+0.058989
498.045	+2	−1	0	0	0	0	3:4	−0.214286
519.551	−2	+3	−1	0	0	0	20:27	−0.060976
568.717	−1	−2	+2	0	0	0	18:25	+0.052265
582.512	0	0	−1	+1	0	0	5:7	+0.059932
590.224	−5	+2	+1	0	0	0	32:45	+0.059761
609.776	+6	−2	−1	0	0	0	45:64	−0.056391
617.488	+1	0	+1	−1	0	0	7:10	−0.056543
680.449	+3	−3	+1	0	0	0	27:40	+0.057471
701.955	−1	+1	0	0	0	0	2:3	+0.272727
729.219	+5	−1	0	−1	0	0	21:32	−0.055703
764.916	+1	−2	0	+1	0	0	9:14	+0.060172
772.627	−4	0	+2	0	0	0	16:25	+0.059524
792.180	+7	−4	0	0	0	0	81:128	−0.056604
813.686	+3	0	−1	0	0	0	5:8	−0.106383
884.359	0	−1	+1	0	0	0	3:5	+0.110294
905.865	−4	+3	0	0	0	0	16:27	+0.083333
933.129	+2	+1	0	−1	0	0	7:12	−0.066879
968.826	−2	0	0	+1	0	0	4:7	+0.081395
996.090	+4	−2	0	0	0	0	9:16	−0.107143
1017.596	0	+2	−1	0	0	0	5:9	−0.085227
1088.269	−3	+1	+1	0	0	0	8:15	+0.082873
1129.328	+4	+1	−2	0	0	0	25:48	−0.051370
1137.039	−1	+3	0	−1	0	0	14:27	−0.051852
1200.000	+1	0	0	0	0	0	1:2	+1.000000

On the rhythmic plane, I also needed a system for determining the priorities of the pulses of a meter. This was achieved by the formulas in Fig.5, which establish a value of metric relevance (termed 'indispensability') for each pulse of a bar stratified as a continued product of primes (e.g. 2x2x3 for $^{12}/_{16}$, of composite length 12) or as a single prime. Both formulas are embedded in each other: the indispensability of any prime p except 2 is based on that of the composite p-1.

Fig.5 – Metric Indispensability Formulas for the n^{th} pulse in a bar of composite and prime length

Metric Indispensability Formula (composite-length bar)

$$\Psi_z(n) = \sum_{r=0}^{z-1}\left\{\prod_{i=0}^{z-r-1} p_i \Psi_{p_{z-r}}\left(1+\left(\left[1+\frac{(n-2) \bmod \prod_{j=1}^{z} p_j}{\prod_{k=0}^{r} p_{z+1-k}}\right] \bmod p_{z-r}\right)\right)\right\}$$

whereby (all variables being whole numbers):

1. $p_0 = p_{z+1} = 1$
2. n is the position in the bar of the pulse in question, starting at 1
3. p_j is the stratification divisor on level j
4. z is the number of levels in the stratification
5. $\Psi_p(x)$ is the Indispensability of the xth pulse of a first-order bar with the prime stratification p
6. $u \bmod v$ is the remainder of the division $(u+mv)/v$, by sufficiently large m never negative
7. $[x]$ is the whole-number component of x

Metric Indispensability Formula (prime-length bar)

If $p=2$, then $\Psi_p(n) = p-n$;

otherwise if $n=p-1$, then $\Psi_p(n) = [p/4]$

or else $\Psi_p(n) = \left[q+2\sqrt{\frac{q+1}{p}}\right]$

whereby

1. n is the position in the bar of the pulse in question, starting at 1
2. $q = \Psi_{p-1}(n-[n/p])$
3. $\Psi_{p-1}(x)$ gives the Indispensabilities for a bar of pulses numbering $p-1$, factorized and stratified with primes in decreasing order of size

See also 5. and 7. in the previous diagram

Here for example are the indispensabilities on the 16^{th}-note level of the meters

$^3/_4$	–	**11**	0	6	3	**9**	1	7	4	**10**	2	**8**	5
$^6/_8$	–	**11**	0	6	2	**8**	4	**10**	1	7	3	**9**	5
$^{12}/_{16}$	–	**11**	0	4	**8**	2	6	**10**	1	5	**9**	3	7

The four most indispensable pulses in each case are shown in bold.

With these methods for quantifying harmony and meter I succeeded in writing my envisaged piano piece *Çoğluotobüsişletmesi* in 1978. These methods were implemented in my computer program *Autobusk* in 1986, with which I composed *variazioni e un pianoforte meccanico* and a series of other pieces, entirely or in part, the most recent of which is *septima de facto* (2006).

ISIS

In 1969 I first doubted the adage that white noise was the 'sum of all frequencies', which implies that even a narrow noise band contains an infinite number of simultaneous frequencies, each of which must have zero amplitude, a model I found unpractical. Stochastic computer programming experience led me to favor the idea that noise could be a single sine wave of rapidly changing frequency. I expected that the frequency probability at any moment of this hypothetically fleeting sine would relate in a simple way to the spectral amplitude of that frequency in the noise.

Soon after MIDI's introduction in 1984, I sent upto 200 MIDI notes per second to an FM-synthesizer, their time-variant pitch probabilities equaling the relative spectral amplitudes of corresponding phoneme frequencies. And indeed, phoneme-generated note-streams, melodies under 20 nps (notes per second), blurred at higher rates (e.g. 200 nps) to pitch clouds timbrally very like the phonemes. I called this *Spectastics*, from 'Spectrally defined Stochastics'.

In 2001 I asked myself: with notes going into the thousands per second, could a tangible pitch be extracted from a single pair of contiguous samples? Indeed yes: with samples as points in a 2D-space bounded vertically by ±1 with time as the x-axis, a sine wave segment connecting any pair of horizontally adjacent points can be shown to have a unique frequency near the sampling rate. Not only is any sound wave interpretable as an ultra-sonic pitch sequence at sample speed, one can also reconstruct a wave acoustically indistinguishable from the original by applying the reverse to the procured pitches. This I called *ISIS*, from 'Intra-Samplar Interpolating Sinusoids'.

Fig.6 shows a set of seven samples (squares on a wavy spline); from these, six frequencies (starting at 43195 Hz) are ISIS-extractable – the samples are mutually joined by sine segments. 'Melodies' of this sort can be transposed down to an audible range and speed. The reverse is also possible: Fig.7 shows J.S.Bach's *Jesu Joy of Man's Desiring* (in cents at left) converted to a sound wave, also seen at left, its spectrum at lower right. The conversion formulas are also given.

Fig.6 – Seven samples as an ISIS-'melody'

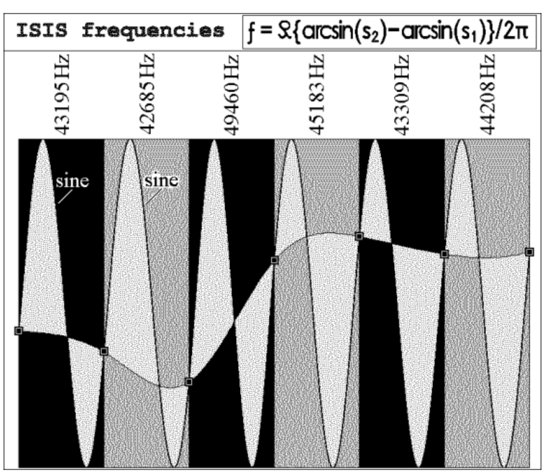
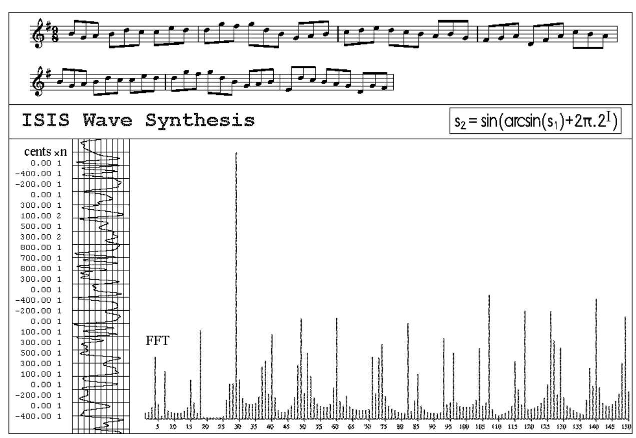


Fig.7 – Bach's *Jesu Joy of Man's Desiring* converted by ISIS into a sound wave (left).



References

Contact the author at <barlow@music.ucsb.edu> for URLs and pdfs.